

# Characterization of the entanglement of two squeezed states

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We study a continuous-variable entangled state composed of two states which are squeezed in two opposite quadratures in phase space. Various entanglement conditions are tested for the entangled squeezed state and we study decoherence models for noise, producing a mixed entangled squeezed state. We briefly describe a probabilistic protocol for entanglement swapping based on the use of this class of entangled states and the main features of a general generation scheme.

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## I. INTRODUCTION

In these years, we have witnessed an increasing interest related to the use of continuous variable (CV) systems as appealing candidates in many applications of quantum information processing (QIP) and quantum communication [1]. For instance, the first experimental observation of CV quantum teleportation [2] has been accomplished, by using a CV entangled channel represented by a simple two-mode squeezed state [3], following an earlier CV-based theoretical proposal [4]. It is now well-understood that CV systems embodied by light field modes may serve as reliable, long-haul information carriers, due to properties of robustness against decoherence [5]. On the other hand, some strategies for the performance of quantum computation with CV systems have been designed in different physical systems [6].

A key element in these QIP tasks is the generation and manipulation of non-classical states of CVs embodying non-classical channels, the paradigmatic examples being represented by coherent superpositions of macroscopically distinguishable coherent states [7, 8] and entangled coherent states [9]. However, it is frequently the case that a full characterization of the non-classical properties of an entangled state of two CV subsystems is made difficult by the lack of objective criteria able to discriminate whether or not a general CV state is entangled. A completely satisfactory answer to this question is possible, so far, only for the special class of Gaussian states, *i.e.* those CV states whose characteristic functions are Gaussian [10, 11]. In this case, necessary and sufficient criteria for entanglement exist [12, 13] which, however, could fail for non-Gaussian states. In order to bypass this bottleneck, very recently some alternative strategies have been designed in order to provide more efficient tests for entanglement in general bipartite CV states [14, 15, 16, 17]. On one hand, the results of this

investigation have the merit of filling the gap between Gaussian and non-Gaussian CV states. On the other hand, they legitimize the theoretical and practical consideration of non-Gaussian states in application of QIP and, by enlarging the class of characterizable CV states, pave the way toward the study of important aspects of quantum entanglement such as its resilience to temperature and mixedness [18].

In this work, we perform a further step along the direction of CV quantum state engineering and characterization by studying a special type of non-Gaussian CV state, namely the entanglement of two single-mode squeezed vacuum states, under the QIP perspective. We address both the cases of pure and mixed states showing that, through the application of some recently suggested tests (some of them being intrinsically operative), entanglement can be revealed regardless of the non-Gaussian nature and the mixedness of the state considered. The aim of our study is to characterize the correlation properties of entangled squeezed states (ESS's) and discuss a possibility of using them in implementation of some basic quantum protocols, such as teleportation [19] and entanglement swapping [20]. In order to complete our study, we also sketch two schemes allowing for the generation of ESS's.

In Section II we introduce the class of ESS's we consider, and formalize the conditional gate we use in order to set entanglement between two initially independent single-mode squeezed states. We then study the properties of these states, both for the pure and the mixed case, by applying some recently suggested tests for entanglement [15, 16, 17]. We quantitatively address the case of thermal- and phase-diffusion noise affecting the purity of the states, quantifying the entanglement via the use of logarithmic negativity [21] and, for a simplified impurity model, sketching the results achieved by applying the entangling power criterion [17]. Section III briefly addresses the application of ESS's in entanglement swapping and teleportation protocols, and finally, Section IV describes two protocols for the generation of ESS's based on interactions of CV modes with a two-level system. Section V summarizes our results.

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## II. CHARACTERIZING THE ENTANGLEMENT OF AN ESS

### A. Characterizing the entanglement in a pure ESS

A vacuum is a minimum uncertainty state whose mean energy is zero. In a phase space representation, as it is well-known, it is possible to modify the uncertainties of the quadrature operators associated with a particular state, still maintaining its minimum-uncertainty nature. More in details, by reducing the uncertainty of one quadrature at the expense of increasing the uncertainty of the canonically conjugate one, a squeezed vacuum is produced with a mean energy which becomes a function of the amount of squeezing. Under a formal point of view, the squeezing operation is accounted by the unitary transformation [3]  $\hat{S}(s) = \exp\left[\frac{s}{2}(a^{\dagger 2} - a^2)\right]$ , where  $s$  is the squeezing parameter and  $a^{\dagger}$  ( $a$ ) is the creation (annihilation) operator of the field. The choice of a direction, in the phase space, for reducing the noise is determined by the argument of  $s$  which, in general, is a complex number. For the sake of convenience, without affecting the generality of our study, here we restrict our attention to the case of a real  $s$ . The state resulting from the squeezing of a initial vacuum state, or a squeezed vacuum, is from now on indicated as  $|\psi_s\rangle = \hat{S}(s)|0\rangle$ . Although squeezed states are now routinely produced, they are usually burdened by noise which alters their minimum-uncertainty nature and makes them intrinsically mixed. As such, in general, they need to be described by a density matrix  $\rho_{s+}$ . Let us consider a two-mode entangled squeezed vacuum (ESV) which, assuming the above conditions of generality, is defined here as

$$\rho_{ESV} = \mathcal{T}(\otimes_{j=a,b} \rho_{s+,j}) \mathcal{T}^{\dagger}, \quad (1)$$

where  $j = a, b$  is a label for the mode considered (mode  $b$  being described by the bosonic operators  $b$  and  $b^{\dagger}$ ), each  $\rho_{s+,j}$  is considered as already normalized and  $\mathcal{T}$  is the unitary transformation

$$\mathcal{T} = \mathbb{1}_a \otimes e^{i\frac{\pi}{2}b^{\dagger}b} + e^{i\phi} e^{i\frac{\pi}{2}a^{\dagger}a} \otimes \mathbb{1}_b. \quad (2)$$

Here  $\mathbb{1}_j$  is the identity operator for mode  $j$ ,  $\phi$  is an arbitrary relative phase and  $e^{i\frac{\pi}{2}j^{\dagger}j}$  describes a phase-shift (by an amount of  $\pi/2$ ) acting on the state of mode  $j$ . In case of pure squeezed vacua, the resulting state can be expressed as

$$|\Psi(\phi)\rangle = \mathcal{N}(|s_+\rangle|s_-\rangle + e^{i\phi}|s_-\rangle|s_+\rangle), \quad (3)$$

where  $\mathcal{N} = 1/\sqrt{2[1 + \text{sech}2s \cos\phi]}$  is the normalization factor and  $|s_{\pm}\rangle = |\psi_{\pm s}\rangle$ . Evidently, being the superposition of two Gaussian states, the ESV,  $|\Psi(\phi)\rangle$ , is not a Gaussian state [22]. It shows some resemblance with the entangled coherent state [9]  $|\Phi_c\rangle = \mathcal{N}_c(|\alpha\rangle - \alpha\rangle + e^{i\phi}|\alpha\rangle - \alpha\rangle)$ , where  $|\alpha\rangle = D(\alpha)|0\rangle$  is a coherent state

of amplitude  $\alpha$  and  $D(\alpha) = e^{(\alpha^*a - \alpha a^{\dagger})}$  is the displacement operator [11]. The resemblance is due to the fact that the overlap  $\langle\alpha| - \alpha\rangle \rightarrow 0$  as  $\alpha \rightarrow \infty$ , making the components of the superposition asymptotically orthogonal. Similarly, we have that  $\langle s_+|s_-\rangle \rightarrow 0$  as  $s \rightarrow \infty$  thus making Eq. (3) the superposition of distinguishable tensorial-product states. The task of our study is the characterization and quantification of the entanglement shared between  $a$  and  $b$  in states (1) and (3).

A well-known sufficient condition for inseparability is the so-called Peres-Horodecki criterion [23] which is based on the observation that the non-completely positive nature of the partial transposition operation may turn an inseparable state into a non-physical state. The signature of this non-physicality, and thus of entanglement, is the appearance of a negative eigenvalue in the eigenspectrum of the partially transposed density matrix of a bipartite system. This approach cannot be directly applied to CV states, spanning infinite dimensional Hilbert space. First attempts to characterize the separability of CV states were focused on second-order moments of some particular quadrature operators [12, 13]. For Gaussian states, whose statistical properties are fully characterized by just second-order moments, this criterion was proven to be necessary and sufficient. Entanglement in non-Gaussian states, which are characterized by the entire hierarchy of statistical moments, may however fail to be detected. This is the reason why entanglement tests for non-Gaussian states are based, in general, on inequalities involving higher-order moments [14] or an infinite hierarchy of inequalities [15, 16]. An alternative to this kind of strategies is represented by the study of the entangling power of the specific non-Gaussian state being considered, *i.e.* the capacity of the state of inducing entanglement (via just local interactions) in an initially separable two-qubit system [17].

In order to start our investigation, we first attempt to analyze the pure state Eq. (3) under the point of view set by the criteria introduced by Simon [12] and Duan *et al.* [13]. As shown by Shchukin and Vogel [15], both these criteria can be expressed in the following elegant forms

$$C_{Simon} = \begin{vmatrix} 1 & \langle a \rangle & \langle a^{\dagger} \rangle & \langle b^{\dagger} \rangle & \langle b \rangle \\ \langle a^{\dagger} \rangle & \langle a^{\dagger 2} \rangle & \langle a^{\dagger}a \rangle & \langle a^{\dagger}b^{\dagger} \rangle & \langle a^{\dagger}b \rangle \\ \langle a \rangle & \langle a^2 \rangle & \langle aa^{\dagger} \rangle & \langle ab^{\dagger} \rangle & \langle ab \rangle \\ \langle b \rangle & \langle ab \rangle & \langle a^{\dagger}b \rangle & \langle b^{\dagger}b \rangle & \langle b^2 \rangle \\ \langle b^{\dagger} \rangle & \langle ab^{\dagger} \rangle & \langle a^{\dagger}b^{\dagger} \rangle & \langle b^{\dagger 2} \rangle & \langle bb^{\dagger} \rangle \end{vmatrix} \geq 0$$

$$C_{Duan} = \begin{vmatrix} 1 & \langle a \rangle & \langle b^{\dagger} \rangle \\ \langle a^{\dagger} \rangle & \langle a^{\dagger}a \rangle & \langle a^{\dagger}b^{\dagger} \rangle \\ \langle b \rangle & \langle ab \rangle & \langle b^{\dagger}b \rangle \end{vmatrix} \geq 0, \quad (4)$$

where  $\langle \mathcal{O} \rangle$  stands for the expectation value of the generic operator  $\mathcal{O}$  calculated over the state being investigated and each inequality is satisfied by separable states. In the specific case of the state  $|\Psi(\phi)\rangle$  in Eq. (3), the mean of the arbitrary operator  $AB$  (with  $A$  and  $B$  general functions  $A = A(a, a^{\dagger})$ ,  $B = B(b, b^{\dagger})$  of each mode's bosonic

operators) is given by

$$\begin{aligned} \langle AB \rangle &= \mathcal{N}^2 (\langle s_+ | A | s_+ \rangle \langle s_- | B | s_- \rangle + \\ &\langle s_- | A | s_- \rangle \langle s_+ | B | s_+ \rangle + 2\text{Re}[e^{i\phi} \langle s_+ | A | s_- \rangle \langle s_- | B | s_+ \rangle]) . \end{aligned} \quad (5)$$

By direct calculation, it is straightforward to conclude that all the first- and second-order moments of  $|\Psi(\phi)\rangle$  vanish, with the exception of

$$\begin{aligned} \langle a^\dagger a \rangle &= 2\mathcal{N}^2 \left( \nu^2 - \frac{\cos(\phi)}{\cosh(2s)} [\nu^2 + \mu\nu \tanh(2s)] \right), \\ \langle b^\dagger b \rangle &= \langle a^\dagger a \rangle = \langle aa^\dagger \rangle - 1 = \langle bb^\dagger \rangle - 1, \end{aligned} \quad (6)$$

where  $\mu = \cosh s$  and  $\nu = \sinh s$ . It can be readily checked, with these tools, that the criteria (4) fail to prove inseparability even though the entanglement properties of the state  $|\Psi(\phi)\rangle$  are out of any doubts. It is indeed possible to construct a discrete two-state basis, following the recipe described in [18], involving the superposition of  $|s_+\rangle$  and  $|s_-\rangle$ . This allows us to reinterpret Eq. (3) as an effective entangled state of two qubits whose entanglement depends, in general, on squeezing parameter  $s$  and relative phase  $\phi$ . In particular, for  $\phi = \pi$ , the entanglement in the effective state is clearly observable as soon as  $s > 0$ .

However, this need not to be seen as a difficult setback since an entangled coherent state [9] also remains undetected in this manner [14]. We simply have to look for a more powerful test. A sufficient criterion based on higher-order moments was proposed by Shchukin and Vogel [15] who observed a useful feature of the determinant of the infinite-dimensional matrix  $M$  composed of the statistical moments

$$\begin{aligned} M_{ij}(\rho^{PT}) &= \langle a^{\dagger i_1} a^{i_2} b^{\dagger i_3} b^{i_4} a^{\dagger j_2} a^{j_1} b^{\dagger j_4} b^{j_3} \rangle^{PT} \\ &= \langle a^{\dagger i_1} a^{i_2} b^{\dagger j_3} b^{j_4} a^{\dagger j_2} a^{j_1} b^{\dagger i_4} b^{i_3} \rangle, \end{aligned} \quad (7)$$

where  $PT$  stands for partial transposition while  $i, j$  are the multi-indices  $i = (i_1, i_2, i_3, i_4)$  and  $j = (j_1, j_2, j_3, j_4)$ . The ordering of the multi-indices is such that  $i > j$  iff

$$\begin{aligned} |i| &> |j| \quad \text{or} \\ |i| &= |j| \quad \text{and} \quad \exists k : i_k > j_k \quad \text{while} \quad i_l = j_l, \forall l > k, \end{aligned} \quad (8)$$

with  $|i| = i_1 + i_2 + i_3 + i_4$  (analogously for  $|j|$ ). Now, the state characterized by a density matrix  $\rho$  is inseparable (*i.e.* has negative partial transpose), if there exists a leading principal minor (or just any principal minor, as was stressed in [24]) for which the relevant determinant is negative. More formally,  $\rho$  is inseparable if

$$\begin{aligned} \exists N = 1, 2, \dots \text{ and } r = (r_1, r_2, \dots, r_N) : \\ \det M^r(\rho^{PT}) < 0, \end{aligned}$$

where  $1 \leq r_1 < r_2 < \dots < r_N$  and  $M^r(\rho^{PT})$  is obtained from  $M(\rho^{PT})$  by deleting all rows and columns except those labelled by  $r_1, \dots, r_N$ . This formalism also includes

other separability criteria. Simon's criterion [12], for instance, is obtained by setting  $r_{\text{Simon}} = (1, 2, 3, 4, 5)$  and Duan *et al.*'s [13] by  $r_{\text{Duan}} = (1, 2, 4)$ . Other criteria can be obtained in a similar way.

The important task in devising a separability criterion for state (3) is therefore finding the right vector  $r$  and considering an appropriate determinant. It turns out that this can indeed be done and the full criterion can be given in the form of the following inequality

$$\begin{vmatrix} 1 & \langle a^\dagger b \rangle & \langle a^\dagger b^\dagger \rangle & \langle ab \rangle & \langle ab^\dagger \rangle \\ \langle ab^\dagger \rangle & \langle aa^\dagger bb^\dagger \rangle & \langle aa^\dagger b^{\dagger 2} \rangle & \langle a^2 bb^\dagger \rangle & \langle a^2 b^{\dagger 2} \rangle \\ \langle ab \rangle & \langle aa^\dagger b^2 \rangle & \langle aa^\dagger b^\dagger b \rangle & \langle a^2 b^2 \rangle & \langle a^2 b^\dagger b \rangle \\ \langle a^\dagger b^\dagger \rangle & \langle a^{\dagger 2} bb^\dagger \rangle & \langle a^{\dagger 2} b^{\dagger 2} \rangle & \langle a^\dagger ab b^\dagger \rangle & \langle a^\dagger ab^{\dagger 2} \rangle \\ \langle a^\dagger b \rangle & \langle a^{\dagger 2} b^2 \rangle & \langle a^{\dagger 2} b^\dagger b \rangle & \langle a^\dagger ab^2 \rangle & \langle a^\dagger ab^\dagger b \rangle \end{vmatrix} < 0. \quad (9)$$

This condition, implying negativity of partial transposition and entanglement, is satisfied for any value of  $\phi$  and  $s$ , although for low values of initial squeezing the violation may be difficult to observe. We have thus designed a proper condition for testing the quantum correlations shared between modes  $a$  and  $b$  in the non-Gaussian state  $|\Psi(\phi)\rangle$ .

As already stated, the use of inequalities based on hierarchies of statistical moments is not the only way we have to infer entanglement in a bipartite non-Gaussian state. Alternatively (and somehow more operatively) we can also check for the capability of the state at hand to induce entanglement, by means of just bi-local interactions, into an initially separable state of two non-interacting qubits. This has been formalized by the so-called entangling power test, which has been applied in order to get some physical insight into some problems of quantum state characterization and has demonstrated to be powerful under many points of view [17, 18, 25]. A full description of the protocol is certainly not the main focus of this work and we refer to the available literature for a detailed analysis of the entangling power test. Here, we just sketch the main points related to the protocol.

We assume that each mode participating to state (3) interacts for a time  $t$  with a qubit initially prepared in its ground state. Each interaction is local to a qubit-mode subsystem. In the spirit of [17, 18, 25], here we assume a resonant Jaynes-Cummings coupling of equal strength  $g$  for each local interaction. The state of the two CV modes is then traced out in order to get the reduced (mixed) density matrix for the qubits. The Peres-Horodecki criterion is then applied to the qubit state. Obviously, if any entanglement is found between the qubits, this can only mean that modes  $a$  and  $b$  were originally entangled, as the local interactions alone can not set entanglement between the qubits. At the same time, the amount of entanglement, here quantified by the logarithmic negativity [21], represents a lower bound to the entanglement shared between the modes [18]. The results are shown in Fig. 1 (a) against the rescaled interaction time  $\tau = gt$ , for  $\phi = 0$  and  $s = 1.1$ . A qualitatively analogous picture could be given for the case of arbitrary values of  $\phi$

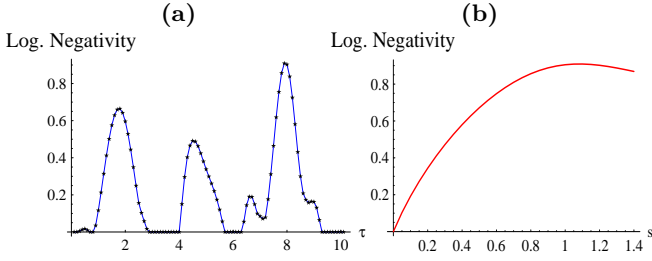


FIG. 1: (Color online) **(a)**: Entangling power as a function of the (dimensionless) rescaled interaction time  $\tau$ , for  $\phi = 0$  and  $s = 1.1$ . **(b)**: Logarithmic negativity against  $s$ , for  $\tau = 8$ .

in Eq. (3). A large amount of transferred entanglement ( $> 0.9$ ) is found to occur at  $\tau \simeq 8$  for this choice of  $s$ , which turns out to be the optimal value for entanglement transfer. The optimality of  $s = 1.1$  is demonstrated in Fig. 1 **(b)**, where the logarithmic negativity is plotted against  $s$ , for  $\tau = 8$ . We have also checked that, for  $\tau \leq 10$ , the transferred entanglement corresponding to other choices of  $s$  and other rescaled interaction times is always smaller than the value achieved at  $\tau = 8$  with  $s = 1.1$ . The considerations of experimental observability of the entangling power discussed in [17, 18, 25] are valid in the context of the present work.

Given the similarity between entangled coherent states and the class of ESV's, it is not difficult to think about applications of ESS's in QIP schemes such as those described in [26]. The large entanglement content (quantitatively demonstrated in the next Subsection), corresponding to specific choices of  $\phi$ , encourages this conjecture. In Section III we investigate their performances as a quantum channel in some paradigmatic protocols of QIP.

Let us now briefly discuss a generalization of the state (3). As a natural extension we consider the state

$$|\Psi'\rangle = \mathcal{N}' (|\alpha_+, \beta_-\rangle + e^{i\phi} |\beta_-, \alpha_+\rangle), \quad (10)$$

where  $\mathcal{N}'$  is the normalization factor and

$$|\alpha_+\rangle = D(\alpha)|s_+\rangle, \quad |\beta_-\rangle = D(\beta)|s_-\rangle \quad (11)$$

are displaced squeezed states. Eq. (10) can be seen as a conjuncture of entangled squeezed states and entangled coherent states [9]. The detailed discussion of the entanglement of such a state is beyond the scope of the present work. However, it is worth discussing some qualitative features which are useful in order to understand the pattern of quantum correlations in Eq. (10). The overlap  $|\langle\alpha_+|\beta_-\rangle|^2$  is critical in determining the entanglement of  $|\Psi'\rangle$ , which increases as the overlap tends to zero. Quantitatively

$$|\langle\alpha_+|\beta_-\rangle|^2 = \frac{1}{\cosh(2r)} \exp\left(-\frac{|\beta - \alpha|^2}{\cosh(2r)}\right), \quad (12)$$

which is a monotonically decreasing function of the difference  $|\beta - \alpha|$  between the single-mode displacements.

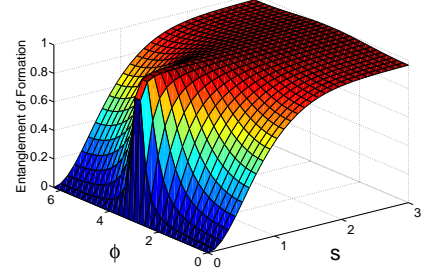


FIG. 2: (Color online) Entanglement of formation with respect to squeezing parameter  $s$  and relative phase  $\phi$ .

The dependence from the squeezing factor  $r$ , on the other hand, is such that the overlap (12) is maximized at  $r = \frac{1}{2} \text{sech}^{-1}(|\beta - \alpha|^{-2})$ . Therefore, there are situations in which an increase of the squeezing can actually lead to the enlargement of the overlap and consequently to the reduction of the entanglement. Note that, in general, state (10) cannot be created by the transformation introduced in Eq. (2) and an *ad hoc* approach theory should be developed. Further generalizations can be achieved by considering the states  $\hat{S}(e^{i\xi_1}|s_1\rangle)|0\rangle$  and  $\hat{S}(e^{i\xi_2}|s_2\rangle)|0\rangle$ , which are squeezed vacua of arbitrary phases, instead of  $|s_{\pm}\rangle$ .

## B. Quantifying the entanglement in a mixed ESS

Having established the existence of entanglement of state (3) we now concentrate on its quantification with respect to pure and mixed ESS's. In general, we will deal with mixed non-Gaussian states for which many proposed entanglement monotones may be proven too difficult to compute and, therefore, unsuitable for our purely demonstrational aims. For this reason, in what follows we will focus on the use of the entanglement measure based on logarithmic negativity, which is computable as

$$LN(\rho) = \log_2 \|\rho^{PT}\|, \quad (13)$$

where the trace norm  $\|A\| = \text{Tr}(\sqrt{A^\dagger A})$  (with  $\text{Tr}$  the trace of a matrix) is calculated as the sum of the absolute values of the eigenvalues of  $A$ . These can be numerically calculated by truncating the dimension of the infinite-dimensional Hilbert space where the ESV is defined to a sufficiently large (but finite) computational subspace.

However, as a consistency check, let us first take advantage of the purity of Eq. (3) and calculate the entanglement of formation [27]

$$E_F = -\text{Tr}[\rho_A \log_2 \rho_A], \quad (14)$$

where  $\rho_A = \text{Tr}_B[\rho]$ . As can be seen in Fig. 2, entanglement is present for all values of  $s$  and  $\phi$ . For  $s \gg 1$ ,  $E_F$  approaches a full ebit, which is clearly the manifestation of a vanishing overlap  $\langle s_+ | s_- \rangle$  (quantitatively,

$\langle s_+ | s_- \rangle \simeq 10^{-2}$ , for  $s = 5$ ). The corresponding effective two-qubit state obtained by using the discretized computational basis (see Subsection II A) is then maximally entangled. However, for  $\phi = \pi$ ,  $E_F$  approaches a full ebit even for small amount of initial squeezing. This behavior can be easily explained as follows. Let us consider the following single-mode states, which are superpositions of states squeezed in two opposite directions

$$\begin{aligned} |\varphi_+\rangle &= \mathcal{N}_{\varphi_+} (|s_+\rangle + |s_-\rangle), \\ |\varphi_-\rangle &= \mathcal{N}_{\varphi_-} (|s_+\rangle - |s_-\rangle), \end{aligned} \quad (15)$$

where  $\mathcal{N}_{\varphi_{\pm}} = \left[ \frac{\sqrt{\cosh s}}{2(\sqrt{\cosh s} \pm 1)} \right]^{1/2}$  are normalization factors. A squeezed vacuum can be represented as a superposition of Fock states as  $|\psi_s\rangle = \frac{1}{\sqrt{\cosh s}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!} \left(-\frac{1}{2} \tanh s\right)^n |2n\rangle$  [3]. With this decomposition, the superposition states take the forms

$$\begin{aligned} |\varphi_+\rangle &= \mathcal{N}_+ \sum_{k=0}^{\infty} \frac{\sqrt{(4k)!}}{(2k)!} \left(\frac{1}{2} \tanh s\right)^{2k} |4k\rangle, \\ |\varphi_-\rangle &= \mathcal{N}_- \sum_{k=0}^{\infty} \frac{\sqrt{(4k+2)!}}{(2k+1)!} \left(\frac{1}{2} \tanh s\right)^{2k+1} |4k+2\rangle. \end{aligned} \quad (16)$$

These are quadro-multiple states which have been discussed by Sanders for the study of superpositions of squeezed states [28]. It is obvious that the two states are orthogonal to each other. Therefore, after normalization, the state  $|\chi\rangle \propto |\varphi_+\rangle|\varphi_-\rangle \pm |\varphi_-\rangle|\varphi_+\rangle$  should bear an ebit. By putting their definitions (15) into  $|\chi\rangle$ , we realize that the states carrying one ebit are exactly the ESV states with  $\phi = \pi$ . Note, that very similar reasoning would be applied explaining the entanglement of the state  $|\Phi(\pi)\rangle \propto |s_+\rangle|s_+\rangle - |s_-\rangle|s_-\rangle$ . This is not surprising as such a state can be obtained from the  $|\Psi(\pi)\rangle$  state by a local  $\pi/2$  phase shift applied to one of the modes.

Let us return to the case of the mixed entangled squeezed state (1) and discuss the influence of imperfection on the entanglement. As a starting point we model a mixed squeezed state by using two types of noise affecting an initial pure squeezed vacuum. The variances of the squeezed (sq) and anti-squeezed (as) quadratures of a squeezed vacuum state affected by thermal Gaussian noise can be written in the form

$$V_{sq} = \frac{1}{2}e^{-2s} + \sigma_{tn}, \quad V_{as} = \frac{1}{2}e^{2s} + \sigma_{tn}, \quad (17)$$

where  $\sigma_{tn}$  characterizes the phase-insensitive Gaussian noise. The density matrix of such a state can be obtained as the classical mixture of randomly displaced squeezed states weighted by a Gaussian distribution of width  $\sigma_{tn}$  as

$$\rho_T = \int \frac{e^{-|\alpha|^2/\sigma_{tn}}}{\pi\sigma_{tn}} D(\alpha) \rho_{s+} D^\dagger(\alpha) d^2\alpha. \quad (18)$$

The second realistic type of noise we consider is caused

by a randomly fluctuating refractive index of the medium through which the initial signal passes in order to generate the ESV's, leading to the density matrix having the form

$$\rho_P = \int \frac{e^{-\varphi^2/2\sigma_{pn}}}{\sqrt{2\pi\sigma_{pn}}} R(\varphi) \rho_{s+} R^\dagger(\varphi) d\varphi. \quad (19)$$

where  $R(\varphi) = \exp(i\varphi a^\dagger a)$  is the phase-shift operator already introduced and we have considered Gaussian fluctuation of the refractive index. The logarithmic negativity corresponding to states (18) and (19) is shown in Figs. 3 and 4, where the entanglement is studied against the relative phase  $\phi$  and the spread of the Gaussian distributions  $\sigma$ . Evidently, as soon as the spread in the Gaussian weight increases, the entanglement diminishes. However, while the logarithmic negativity becomes zero for most of the values of  $\phi$  for  $\sigma_{pn} \geq 1$  in the case of phase noise, it is still as large as 0.1 for  $\sigma_{tn} \simeq 2$ , uniformly with respect to  $\phi$ . An analogous decrease in the entanglement can be found by considering again the entangling power test. In this case, in order to simplify the (otherwise computationally prohibitive) calculations, the mixedness of the ESV is artificially modelled as follows. We consider a single-mode squeezed state  $|\psi_s\rangle$  and an ancillary vacuum state mixed at a beam splitter (BS) of transmissivity  $\cos^2 \theta$ . By tracing out the output ancillary state, we get an output mixed squeezed state whose purity depends on the transmissivity of the BS. This models each of the  $\rho_{s+,j}$  entering Eq. (1). By applying the entangling power test to the resulting state, we have quantitatively found a transferred entanglement no larger than 0.4 for the optimal squeezing  $s = 1.1$  and  $\phi = 0$ .

### III. ENTANGLEMENT SWAPPING AND TELEPORTATION

In this Section we briefly discuss the possibility of implementing some basic protocols for QIP by using the class of ESS's. In particular, as a prototypical example

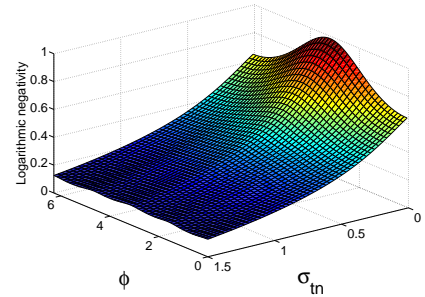


FIG. 3: (Color online) Logarithmic negativity of entangled state of squeezed states with thermal noise, against the relative phase  $\phi$  and the width of the Gaussian weight  $\sigma$ . The initial squeezing considered is  $s = 1$ .

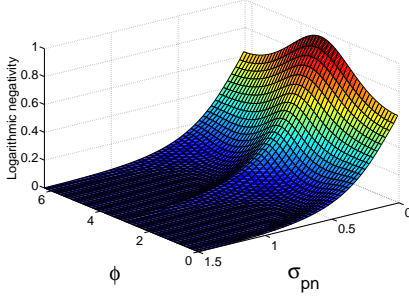


FIG. 4: (Color online) Logarithmic negativity of ESV affected by phase-diffusion noise, against the relative phase  $\phi$  and the width of the Gaussian weight  $\sigma$ . The initial squeezing considered is  $s = 1$ .

where the entanglement of a channel plays a crucial role, we address the probabilistic realization of entanglement swapping. We also show how a quantum teleportation protocol can be realized.

Starting with two entangled subsystems embodied by the two pairs of modes 1 and 2 and 3 and 4, the aim of entanglement swapping is to create an entangled state of modes 1 and 4. This can be performed by a joint measurement of modes 2 and 3 and either post-selection or local operations performed on remaining modes [20]. Let us consider the initial states for the two subsystems to be the tensorial product of the two ESV's

$$\begin{aligned} |\Psi(\pi)\rangle_{12} &= \mathcal{N}(|s_+\rangle|s_-\rangle - |s_-\rangle|s_+\rangle)_{12}, \\ |\Phi(\pi)\rangle_{34} &= \mathcal{N}(|s_+\rangle|s_+\rangle - |s_-\rangle|s_-\rangle)_{34}. \end{aligned} \quad (20)$$

As stated above, the task is to create one ebit of entanglement within the subsystem 1 and 4. In order to achieve this, the modes 2 and 3 are superimposed on a balanced BS realizing the transformations  $a_2 \rightarrow (a_2 + a_3)/\sqrt{2}$  and  $a_3 \rightarrow (a_3 - a_2)/\sqrt{2}$ . Consequently, the initial state  $|\Psi(\pi)\rangle_{12} \otimes |\Phi(\pi)\rangle_{34}$  becomes

$$\begin{aligned} \mathcal{N}^2 |\psi\rangle_{1234} &= \\ & S_{23}(s)|0,0\rangle_{23}|s_-,s_-\rangle_{14} \\ & + S_2(-s)S_3(-s)|0,0\rangle_{23}|s_+,s_-\rangle_{14} \\ & + S_2(s)S_3(s)|0,0\rangle_{23}|s_-,s_+\rangle_{14} \\ & + S_{23}(-s)|0,0\rangle_{23}|s_+,s_+\rangle_{14}, \end{aligned} \quad (21)$$

where  $S_{ab}(s)|0,0\rangle_{ab} = (\cosh s)^{-1} \sum_{n=0}^{\infty} (\tanh s)^n |n,n\rangle_{ab}$  is a two-mode squeezed vacuum state. The second and third terms in Eq. (21) contain only even number of photons in modes 2 and 3 while the first and last terms contain both even and odd numbers. The entanglement swapping can now be achieved by performing the projection onto odd photon-number states of modes 2 and 3 formally described by  $\Pi_{23} = \sum_k |2k+1\rangle_{23}\langle 2k+1|$  and tracing over these modes. This leads to the output state

$$\left[ \mathcal{N}^2 \sum_{k=0}^{\infty} \frac{(\tanh s)^{2+4k}}{(\cosh s)^2} \right] |\Phi(\pi)\rangle_{14} \langle \Phi(\pi)|. \quad (22)$$

The desired entangled state of modes 1 and 4 is obtained with a probability which can be easily evaluated to be  $1/4$ . A probabilistic entanglement swapping is possible by using two entangled resources embodied in ESV's.

A similar approach can be used in order to realize quantum teleportation [19]. Let us assume that our task is the teleportation between two spatially separated nodes  $A$  and  $B$  of an unknown state of an input mode  $I$   $(\alpha|s_+\rangle + \beta|s_-\rangle)_I$ . We consider the state  $|\Phi(\pi)\rangle_{12}$  as an entangled resource shared by  $A$  and  $B$ . At the local node  $A$ , the input mode  $I$  and mode 1 of the entangled state are mixed on a BS and a projection analogous to the one described in the entanglement swapping protocol is performed. By projecting onto odd photon-number states of the subsystem  $I$  and 1, a classical message is sent to node  $B$  which holds mode 2. At this stage, a  $\pi/2$  rotation in phase space on mode 2 allows for the reconstruction of the unknown input state at node  $B$ . Analogously to the entanglement swapping, the entire procedure succeeds with a probability equal to  $1/4$ .

#### IV. GENERATION OF ESS

We complete our characterization of ESS's by briefly addressing the problem of generating this class of states. General recipes can be given for the realization of an ESS, both exploiting similar physical mechanisms. In details, in order to create ESS's we require the evolution described by the operator  $\mathcal{T}$ . This can be accomplished in several ways, two of them depicted on Figs. 5, both requiring coupling of two CV systems with an ancillary two-level system. In both the schemes, the squeezed states need not to be pure, although we are going to assume this quality in order to simplify the description. The most straightforward way employs a pair of identical single mode squeezed states that are coupled to the ancillary system via controlled phase-flip operations acting as

$$\begin{aligned} \Theta_{\pi/2}^0 |s_+\rangle|0\rangle &= |s_-\rangle|0\rangle, & \Theta_{\pi/2}^0 |s_+\rangle|1\rangle &= |s_+\rangle|1\rangle, \\ \Theta_{\pi/2}^1 |s_+\rangle|0\rangle &= |s_+\rangle|0\rangle, & \Theta_{\pi/2}^1 |s_+\rangle|1\rangle &= |s_-\rangle|1\rangle, \end{aligned} \quad (23)$$

where  $|0\rangle$  and  $|1\rangle$  are the states of the computational basis of the ancillary qubit (cf. Fig. 5 (a)) [8].

The second possibility is to use a two-mode squeezed vacuum  $S_{12}$  as a resource. In this case, it is sufficient to perform a single controlled phase-shift, by an amount of  $\pi$ , on the joint state of an ancilla and one of modes  $j = 1, 2$ , followed by a mixing of modes 1 and 2 using a balanced BS (cf. Fig. 5 (b)). Notice however that in the first scheme we do not need to prepare the input modes in independent squeezed states. They can be obtained by using a two-mode squeezed state whose components are previously mixed at a balanced BS. In this way, a single squeezed resource is exhausted in the generation of an ESS, in both the schemes.

Given an arbitrary ancillary state of the form  $\alpha|0\rangle + \beta|1\rangle$  ( $|\alpha|^2 + |\beta|^2 = 1$ ), both these possibilities lead to a



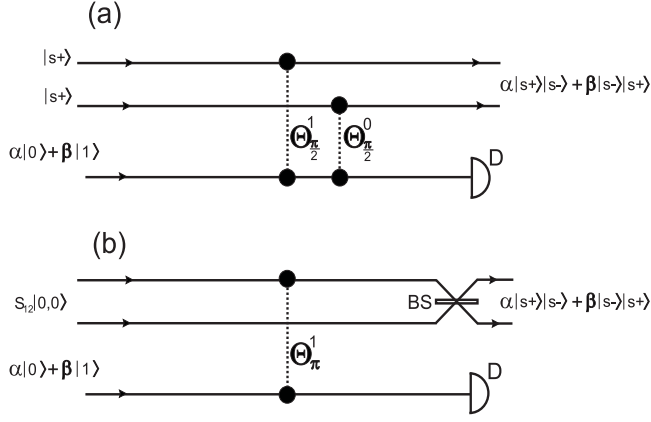


FIG. 5: Generation schemes for ESS's. The vertical dotted lines represent the controlled phase-flip operations, D stands for a detector measuring in the  $|\pm\rangle$  basis and BS is a balanced beam splitter.

joint state of the three participating modes reading

$$\alpha|s_+\rangle|s_-\rangle|0\rangle + \beta|s_-\rangle|s_+\rangle|1\rangle. \quad (24)$$

The ancillary qubit is now measured in the rotated basis  $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ ,  $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$  and, conditioned on the outcome of the measurement, the resulting two-mode CV states read

$$\alpha|s_+\rangle|s_-\rangle \pm \beta|s_-\rangle|s_+\rangle. \quad (25)$$

The  $+$  ( $-$ ) sign corresponds to the ancillary state  $|+\rangle$  ( $|-\rangle$ ) being measured. The generality of the described protocols allows for practical implementations in various physical setups.

The key part in the discussed setups is the cross-Kerr type interaction  $\Theta_\gamma = e^{i\gamma a^\dagger ab^\dagger b}$  of the ancilla qubit with the  $j$ -th mode of interest. This interaction plays a critical role in the engineering of non-classical CV states like superpositions of coherent states and entangled coherent states and in quantum computation with coherent states [8, 29]. The generation of ESS's requires either two cross-Kerr interactions with parameters  $\gamma = \pi/2$  (for the case of the scheme presented in Fig. 5 (a)), or a single one with  $\gamma = \pi$  (Fig. 5 (b)). It can be easily checked that in both cases, as a result of the photon-distribution properties of squeezed states, the required effective rate of nonlinearity is half the analogous nonlinearity needed for the generation of entangled coherent states (by using the setup in Figs. 5).

In a setup of cavity-quantum electrodynamics (cavity-QED), the scheme in Fig. 5 (a) can be realized with the ancillary qubit being embodied by a flying two-level atom, which passes through a microwave cavity supporting a single-mode field [30]. If the atom-field interaction is set to be far off-resonant (with a large detuning being set by the ac Stark shift effect on the atomic level spacing induced by a static intracavity electric field), the effective time-evolution operator regulating the dynamics of the

joint atom-field state, is of the required cross-Kerr type. Due to the long lifetimes of both the flying qubit and the cavity field [31], the interaction time leading to  $\gamma \sim \pi$  can be achieved with current technology [26, 31]. On the other hand, still within cavity-QED, the required coupling responsible for the conditioned phase-flip in Fig. 5 (b) can be accomplished by arranging an interaction between one mode of the two-mode squeezed state with a single atom trapped inside an optical cavity, as in the scheme described by Wang and Duan [32]. In this case, one can take an advantage of the recently achieved regime of strong interaction between the cavity field and the trapped atom where the conditioned phase-flip can be obtained within the coherence time of both the atom and the cavity (see Wang and Duan in [32] for a detailed discussion).

An all-optical implementation scheme would require a path encoded photonic qubit as an ancilla. This can be straightforwardly achieved using the polarization of a photon and a polarization beam splitter. The cross-Kerr coupling could then be implemented in doped optical fibres. However, the strength of Kerr nonlinearity is very low and makes the approach unrealistic for the current state of the art [33]. A more promising approach relies on the use of double-electromagnetically induced transparency (double-EIT) [34]. Here, the ancillary light field and the target mode are present in an effective highly-nonlinear medium where the group velocities of the two fields are simultaneously reduced (by the EIT mechanism). If the effectively nonlinear medium is characterized by a sufficient degree of symmetry, the two velocities will be of the same (small) value. Consequently, the interaction time between two light fields can be increased in a way to achieve  $\gamma \sim \pi$ . In these years, various energy configurations of the effective nonlinear medium and many coupling schemes with light have been considered, in order to achieve a perfect double-EIT, free from residual absorption (which limits the achievable nonlinear coupling rate), Doppler-broadening (which spoils the transparency window induced by EIT) and dephasing. The desired phase shift can be then tuned by a proper choice of the strength and length of the signal pulse. With current technology, a single photon induced phase shift of  $\approx \pi/12$  can be expected [35].

## V. REMARKS

In the important context of quantum state engineering and characterization, we have studied the entanglement properties of the special class of ESS's. The bottleneck set by the intrinsic non-Gaussianity of the states considered is bypassed by considering an entanglement criterion based on higher-order statistical moments and by the application of the entangling power test. Clear physical interpretations for the entanglement found for different values of the relative phase in an ESS have been provided. This helps the comprehension of the quantitative results

achieved by the use of the logarithmic negativity. As examples for applications of an ESS, we have suggested two probabilistic schemes for entanglement swapping and teleportation. Finally, we have briefly described a general generation scheme based on the use of a single ancilla and non-linear interactions whose practical implementation can be found in various physical setups.

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